

Mutant knots with symmetry

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Abstract

Mutant knots, in the sense of Conway, are known to share the same Homfly polynomial. Their 2-string satellites also share the same Homfly polynomial, but in general their m -string satellites can have different Homfly polynomials for $m > 2$. We show that, under conditions of extra symmetry on the constituent 2-tangles, the directed m -string satellites of mutants share the same Homfly polynomial for $m < 6$ in general, and for all choices of m when the satellite is based on a cable knot pattern.

We give examples of mutants with extra symmetry whose Homfly polynomials of some 6-string satellites are different, by comparing their quantum $sl(3)$ invariants.

1 Introduction

This paper has been inspired by recent observations of Ochiai and Jun Murakami about the Homfly skein theory of m -parallels of certain symmetrical 2-tangles. In [8] Ochiai remarks that the 3-parallels of the tangle AB in figure 1 and its mirror image $\overline{AB} = BA$ are equal in the Homfly skein of 6-tangles, in other words, in the Hecke algebra H_6 , [1].

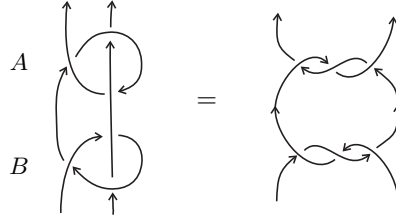


Figure 1:

As a consequence, the 3-parallels of any mutant pair of knots given by composing the 2-tangles AB and BA with any other 2-tangle C and then closing will share the same Homfly polynomial.

This is in contrast with the known fact that 3-parallels of mutant knots in general can have different Homfly polynomials, [7, 4].

There is interest in the extent to which the Homfly polynomial of m -parallels or other m -string satellites can distinguish mutants which are closures of ABC and BAC with A and B as above. Ochiai has found that the 4-parallels of AB and BA are different in the skein H_8 .

The purpose of this paper is to show that if A and B are any two oriented 2-tangles with symmetry

$$A = \begin{array}{c} \boxed{A} \\ \downarrow \curvearrowright \end{array}, \quad B = \begin{array}{c} \boxed{B} \\ \downarrow \curvearrowright \end{array}$$

then the m -parallels, and indeed any directed m -string satellite, of knots \widehat{ABC} and \widehat{BAC} shown in figure 2 share the same Homfly polynomial for $m < 6$.

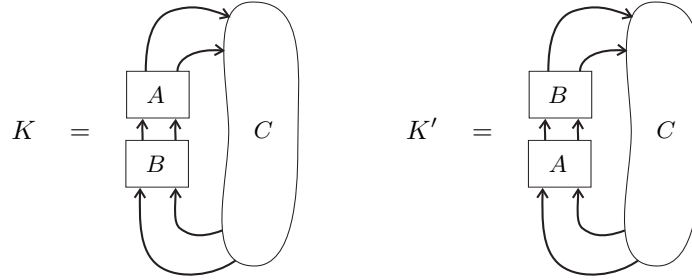


Figure 2: Tangle interchange

In contrast there exist examples of A, B and C , including Ochiai's case with

$$A = \begin{array}{c} \uparrow \uparrow \\ \curvearrowright \end{array}, \quad B = \begin{array}{c} \uparrow \uparrow \\ \curvearrowleft \end{array},$$

for which the Homfly polynomials of the 6-fold parallel are different.

As an unexpected extension of the main result we show that the Homfly polynomial of a genuine connected cable, based on the (m, n) torus knot pattern, with m and n coprime, for any number of strings, m , will not distinguish mutants with symmetry above, although a more general connected satellite pattern can do so.

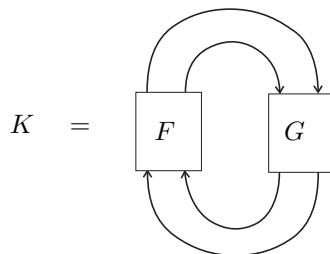
The examples which exhibit differences for the directly oriented 6-parallel can also be used to show that the 4-parallels with two pairs of reverse strands have distinct Homfly polynomials.

The proofs are based on the relation of the Homfly satellite invariants to quantum $sl(N)$ invariants, and the techniques are an extension of work with Cromwell [4] and with H. Ryder [6]. The eventual calculations that exhibit the difference of invariants in the specific example depend on the 27 dimensional irreducible module over $sl(3)$ corresponding to the partition 4, 2, and some Maple calculations following similar lines to those in [6].

2 Shared invariants of mutants

The term *mutant* was coined by Conway, and refers to the following general construction.

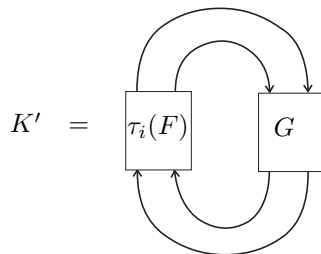
Suppose that a knot K can be decomposed into two oriented 2-tangles F and G



A new knot K' can be formed by replacing the tangle F with the tangle $F' = \tau_i(F)$ given by rotating F through π in one of three ways,

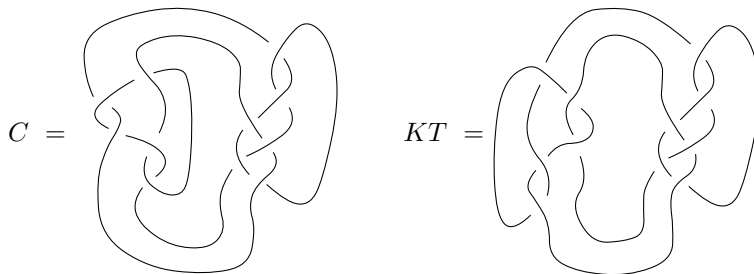
$$\tau_1(F) = \boxed{F} \begin{array}{c} \searrow \\ \curvearrowright \\ \swarrow \end{array}, \quad \tau_2(F) = \boxed{F} \begin{array}{c} \nearrow \\ \curvearrowright \\ \searrow \end{array}, \quad \tau_3(F) = \boxed{F} \begin{array}{c} \downarrow \\ \curvearrowright \\ \uparrow \end{array},$$

reversing its string orientations if necessary. Any of the three knots



is called a *mutant* of K .

The two 11-crossing knots, C and KT , with trivial Alexander polynomial found by Conway and Kinoshita-Terasaka are the best-known example of mutant knots.



2.1 Satellites

A satellite of K is determined by choosing a diagram Q in the standard annulus, and then drawing Q on the annular neighbourhood of K determined by the framing, to give the satellite knot $K * Q$. We refer to this construction as *decorating K with the pattern Q* , as shown in figure 3.

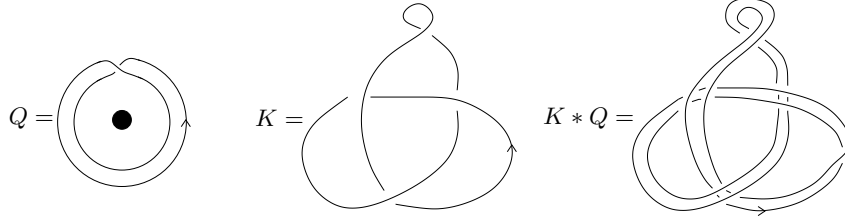


Figure 3: Satellite construction

For fixed Q the Homfly polynomial $P(K * Q)$ of the satellite is an invariant of the framed knot K . The invariants $P(K * Q)$ as Q varies make up the *Homfly satellite invariants* of K . We use the alternate notation $P(K; Q)$ in place of $P(K * Q)$ when we want to emphasise the dependence on K .

The general symmetry result compares the invariants of two knots K and K' made up of 2-tangles A , B and C , by interchanging A and B as in figure 2.

Theorem 1. *Suppose that A and B are both symmetric under the half-twist τ_3 , so that*

$$A = \begin{array}{c} \boxed{A} \\ \hookrightarrow \end{array}, \quad B = \begin{array}{c} \boxed{B} \\ \hookrightarrow \end{array}$$

*Let K and K' be knots which are the closure of ABC and BAC respectively for any tangle C , as in figure 2. Then $P(K * Q) = P(K' * Q)$ for every closed braid pattern Q on $m < 6$ strings.*

Remark 1. *Our proof will apply equally to the case where Q is the closure of a directly oriented m -tangle with $m < 6$.*

In order to prove the theorem we must rewrite the Homfly satellite invariants in terms of quantum $sl(N)$ invariants, so we now give a brief summary of the relations between these invariants, originally established by Wenzl. Further details can be found in [1] and the thesis of Lukac, [3], including details of variant Homfly skeins with a framing correction factor, x . These are isomorphic to the skeins used here but the parameter allows a careful adjustment of the quadratic skein relation to agree directly with the natural relation arising from use of the quantum groups $sl(N)$.

2.2 Homfly skeins

For a surface F with some designated input and output boundary points the (linear) Homfly skein of F is defined as linear combinations of oriented diagrams in F , up to Reidemeister moves II and III, modulo the skein relations

$$\begin{aligned} 1. \quad & \begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} - \begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array} = (s - s^{-1}) \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \nwarrow \\ \nearrow \end{array}, \\ 2. \quad & \begin{array}{c} \uparrow \\ \circlearrowleft \end{array} = v^{-1} \begin{array}{c} \uparrow \\ \int \end{array}. \end{aligned}$$

It is an immediate consequence that

$$\begin{array}{c} \circlearrowleft \\ \int \end{array} = \delta \begin{array}{c} \uparrow \\ \int \end{array},$$

where $\delta = \frac{v^{-1} - v}{s - s^{-1}} \in \Lambda$. The coefficient ring Λ is taken as $Z[v^{\pm 1}, s^{\pm 1}]$, with denominators $s^r - s^{-r}$, $r \geq 1$.

The skein of the annulus is denoted by \mathcal{C} . It becomes a commutative algebra with a product induced by placing one annulus outside another.

The skein of the rectangle with m inputs at the top and m outputs at the bottom is denoted by H_m . We define a product in H_m by stacking one rectangle above the other, obtaining the Hecke algebra $H_m(z)$, when $z = s - s^{-1}$ and the coefficients are extended to Λ . The Hecke algebra H_m can also be regarded as the group algebra of Artin's braid group B_m generated by the elementary braids σ_i , $i = 1, \dots, m-1$, modulo the further quadratic relation $\sigma_i^2 = z\sigma_i + 1$.

The closure map from H_m to \mathcal{C} is the Λ -linear map induced by mapping a tangle T to its closure \hat{T} in the annulus (see figure 4). We refer to a diagram $Q = \hat{T}$ as a *directly oriented pattern*.

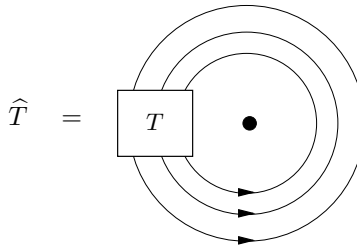


Figure 4: The closure map

The image of this map is denoted by \mathcal{C}_m , which has a useful interpretation as the space of symmetric polynomials of degree m in variables x_1, \dots, x_N for large enough N . Moreover, the submodule $\mathcal{C}_+ \subset \mathcal{C}$ spanned by the union $\cup_{m \geq 0} \mathcal{C}_m$ is a subalgebra of \mathcal{C} isomorphic to the algebra of the symmetric functions.

2.3 Quantum invariants

A quantum group \mathcal{G} is an algebra over a formal power series ring $\mathbf{Q}[[\hbar]]$, typically a deformed version of a classical Lie algebra. We write $q = e^\hbar, s = e^{\hbar/2}$ when working in $sl(N)_q$. A finite dimensional module over \mathcal{G} is a linear space on which \mathcal{G} acts.

Crucially, \mathcal{G} has a coproduct Δ which ensures that the tensor product $V \otimes W$ of two modules is also a module. It also has a *universal R-matrix* (in a completion of $\mathcal{G} \otimes \mathcal{G}$) which determines a well-behaved module isomorphism

$$R_{VW} : V \otimes W \rightarrow W \otimes V.$$

This has a diagrammatic view indicating its use in converting coloured tangles to module homomorphisms.

A braid β on m strings with permutation $\pi \in S_m$ and a colouring of the strings by modules V_1, \dots, V_m leads to a module homomorphism

$$J_\beta : V_1 \otimes \dots \otimes V_m \rightarrow V_{\pi(1)} \otimes \dots \otimes V_{\pi(m)}$$

using $R_{V_i, V_j}^{\pm 1}$ at each elementary braid crossing. The homomorphism J_β depends *only on the braid β itself*, not its decomposition into crossings, by the Yang-Baxter relation for the universal R -matrix.

When $V_i = V$ for all i we get a module homomorphism $J_\beta : W \rightarrow W$, where $W = V^{\otimes m}$. Equally, a directed m -tangle T determines an endomorphism J_T of $W = V^{\otimes m}$. Now any $sl(N)$ module W decomposes as a direct sum $\bigoplus (W_\mu \otimes V_\mu^{(N)})$, where W_μ is the linear subspace consisting of the *highest weight vectors* of type μ associated to the module $V_\mu^{(N)}$. Highest weight subspaces of each type are preserved by module homomorphisms, and so J_T determines (and is determined by) the restrictions $J_T(\mu) : W_\mu \rightarrow W_\mu$ for each μ .

If a knot K is decorated by a pattern Q which is the closure of an m -tangle T then its quantum invariant $J(K * Q; V)$ can be found from the endomorphism J_T of $W = V^{\otimes m}$ in terms of the quantum invariants of K and the highest weight maps $J_T(\mu) : W_\mu \rightarrow W_\mu$ by the formula

$$J(K * Q; V) = \sum c_\mu J(K; V_\mu^{(N)}) \quad (1)$$

with $c_\mu = \text{tr } J_T(\mu)$. This formula follows from lemma II.4.4 in Turaev's book [11]. Here μ runs over partitions with at most N parts when we are working with $sl(N)$, and we set $c_\mu = 0$ when W has no highest weight vectors of type μ .

Proof of theorem 1. Take $V = V^{(N)}$ as the fundamental module of dimension N for $sl(N)$. Then the only highest weight types μ which occur in equation (1)

are partitions of m with at most N rows. Because $J(K * Q; V^{(N)}) = P(K * Q)$ when $v = s^{-N}$ we can show that $P(K * Q) = P(K' * Q)$ by showing that $J(K * Q; V^{(N)}) = J(K' * Q; V^{(N)})$ for all N . By equation 1 it is then enough to show that $J(K; V_\mu^{(N)}) = J(K'; V_\mu^{(N)})$ for all N and all partitions $\mu \vdash m$.

Now each tangle A and B determines an endomorphism J_A, J_B of $V_\mu \otimes V_\mu$. If J_A and J_B commute then $J(K; V_\mu) = J(K'; V_\mu)$. The endomorphisms J_A and J_B are determined by their restriction $J_A(\nu), J_B(\nu)$ to the highest weight subspaces W_ν in the decomposition $V_\mu \otimes V_\mu = \sum W_\nu \otimes V_\nu$, so it is enough to show that $J_A(\nu)$ and $J_B(\nu)$ commute where V_ν is a summand of $V_\mu \otimes V_\mu$. This is certainly the case for all ν where W_ν is 1-dimensional, which includes the case of single row or column partitions μ , [4].

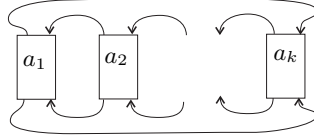
As a special case of the work of Rosso and Jones, [9, 5], we know that the endomorphism of $V_\mu \otimes V_\mu$ for the full twist Δ^2 on two strings operates as a scalar $e^{f(\nu)}$ on each highest weight space W_ν , while the half twist Δ , represented by the R -matrix $R_{V_\mu V_\mu}$, operates on W_ν with two eigenvalues $\pm e^{\frac{1}{2}f(\nu)}$.

The positive and negative eigenspaces correspond to the classical decomposition of the Schur function $(s_\mu)^2$ into symmetric and skew-symmetric parts, $h_2(s_\mu)$ and $e_2(s_\mu)$, and the dimension of each eigenspace of W_ν is the multiplicity of s_ν in $h_2(s_\mu)$ and $e_2(s_\mu)$ respectively.

Now $A = \tau_3(A)$, so that $A\Delta = \Delta A$. Hence the endomorphism J_A , and similarly J_B , preserves the positive and negative eigenspaces of each W_ν . If these eigenspaces have dimension 1 or 0 then J_A and J_B will commute on W_ν .

The theorem is then established by checking that no s_ν occurs in $h_2(s_\mu)$ or $e_2(s_\mu)$ with multiplicity > 1 for any μ with $|\mu| \leq 5$. The decomposition of all of these can be quickly confirmed using the Maple program SF of Stembridge [10]. \square

Corollary 2. *Examples include k -pretzel knots $K(a_1, \dots, a_k)$ with odd a_i .*



Here the numbers a_i can be permuted without changing the Homfly polynomial of any satellite with ≤ 5 -strings.

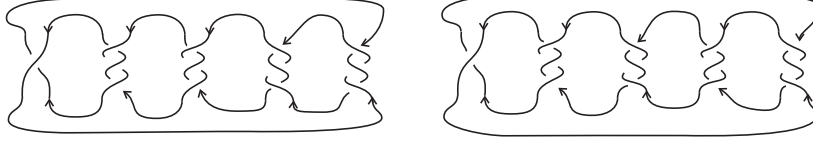
3 Satellites with different Homfly polynomials

A further check with the program SF when $|\mu| = 6$ shows that there are just three partitions, $\mu = 4, 2$, its conjugate $\mu = 2, 2, 1, 1$ and $\mu = 3, 2, 1$ whose symmetric square $h_2[s_\mu]$ contains summands with multiplicity > 1 , as does the exterior squares of $\mu = 3, 2, 1$. Explicitly $h_2[s_{4,2}] = s_{8,4} + s_{8,2,2} + s_{7,4,1} + s_{7,3,2} + s_{7,3,1,1} + s_{6,6} + s_{6,5,1} + 2s_{6,4,2} + s_{6,3,2,1} + s_{6,2,2,2} + s_{5,5,1,1} + s_{5,4,3} + s_{5,4,2,1} + s_{5,3,3,1} + s_{4,4,4} + s_{4,4,2,2}$. This means that, although m -string satellites of K

and K' must share the Homfly polynomial when $m \leq 5$, it is possible for the Homfly polynomials of some 6-string satellites to differ.

We give an example now where this does indeed happen.

Theorem 3. *Let K and K' be the pretzel knots $K = K(1, 3, 3, -3, -3)$ and $K' = K(1, 3, -3, 3, -3)$.*



*The 6-fold parallels $K * Q$ and $K' * Q$, where Q is the closure of the identity braid on 6 strings, have different Homfly polynomials.*

Proof. Write K and K' as the closure of the products $\Delta ABAB$ and $\Delta BAAB$ respectively, where

$$A = \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array} \quad \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array}, \quad B = \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array} \quad \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \\ \uparrow \end{array},$$

are the partially closed 3-braids shown, and Δ is the positive half-twist. We show that $P(K * Q) \neq P(K' * Q)$ when $v = s^{-3}$. These values are given by the $sl(3)$ quantum invariants $J(K * Q; V^{(3)})$ and $J(K' * Q; V^{(3)})$, where $V^{(3)}$ is the fundamental 3-dimensional module for $sl(3)$. Since Q is the closure of the identity braid on 6 strings it induces the identity endomorphism on the module $(V^{(3)})^{\otimes 6}$. This module decomposes as $\bigoplus W_\mu \otimes V_\mu^{(3)}$ where μ runs through partitions of 6 with at most 3 rows. The trace of the identity on W_μ is just $d_\mu = \dim W_\mu$, giving

$$J(K * Q; V^{(3)}) = \sum d_\mu J(K; V_\mu^{(3)}).$$

The only partition μ in this range for which the exterior or symmetric square contains highest weight vectors of multiplicity > 1 is the partition $\mu = 4, 2$, since the partition $\mu = 2, 2, 1, 1$ has 4 rows and the repeated factors for $\mu = 3, 2, 1$ occur for partitions with more than 3 rows. Now $J_A(\mu)J_B(\mu) = J_B(\mu)J_A(\mu)$ for all other μ since A and B are symmetric up to altering the framing on both strings, while maintaining the writhe. Then

$$P(K * Q) - P(K' * Q) = d_\mu (J(K; V_\mu^{(3)}) - J(K'; V_\mu^{(3)}))$$

when $v = s^{-3}$ and $\mu = 4, 2$. Since $d_\mu \neq 0$ it is enough to show that $J(K; V_\mu^{(3)}) \neq J(K'; V_\mu^{(3)})$. The module $V_\mu^{(3)}$ has dimension 27.

We now work in the quantum group $sl(3)$ and drop the superscript (3) from the irreducible modules.

Decompose the module $V_\mu \otimes V_\mu$ as $\sum W_\nu \otimes V_\nu$ and compare the endomorphisms given by the tangles $T = ABAB\Delta$ and $T' = BAAB\Delta$.

In this case just one of the invariant subspaces of highest weight vectors has dimension > 1 . It can be shown that the corresponding 2×2 matrices A_μ and B_μ arising from the two mirror-image tangles A and B with 3 crossings satisfy $\text{tr}(A_\mu B_\mu A_\mu B_\mu - A_\mu A_\mu B_\mu B_\mu) \neq 0$, which results in a difference in their $sl(3)$ invariants $J(K; V_\lambda)$.

None of the other 6-cell invariants differ on the two knots. Consequently the 6-parallelisms have different $sl(3)$ invariants. The $sl(3)$ invariant of the 6-parallelisms of the two pretzel knots coloured with the fundamental module, and thus their Homfly polynomials, are then different. \square

3.1 Use of the quantum group $sl(3)_q$

The calculation of the 2×2 matrices A_ν and B_ν giving the effect of the two tangles on the highest weight vectors where there is a 2-dimensional highest weight subspace of the symmetric part of the module depends on finding the explicit action of the quantum group on the 27-dimensional module $V_\mu^{(3)}$ with $\mu = 4, 2$ and its tensor square, as well as the homomorphism representing its R -matrix. I used the linear algebra packages in Maple to handle the matrix working and subsequent polynomial factorisation, following fairly closely the techniques developed with H. Ryder in the paper [6].

In the interests of reproducibility I give an account of the methods used, and some of the checks applied during the calculations, to test against known properties.

We start from a presentation of the quantum group $sl(3)_q$ as an algebra with six generators, $X_1^\pm, X_2^\pm, H_1, H_2$, and a description of the comultiplication and antipode.

Let M be any finite-dimensional left module over $sl(3)_q$. The action of any one of these six generators Y will determine a linear endomorphism Y_M of M . We build up explicit matrices for these endomorphisms on a selection of low-dimensional modules, using the comultiplication to deal with the tensor product of two known modules, and the antipode to construct the action on the linear dual of a known module. We must eventually determine the matrices Y_M for our module $M = V_{\square\square\square}$, and find the 729×729 R -matrix, R_{MM} which represents the endomorphism of $M \otimes M$ needed for crossings.

We follow Kassel in the basic description of the quantum group from using generators H_1 and H_2 for the Cartan sub-algebra, but with generators X_i^\pm in place of X_i and Y_i . We use the notation $K_i = \exp(hH_i/4)$, and set $a = \exp(h/4)$, $s = \exp(h/2) = a^2$ and $q = \exp(h) = s^2$, unlike Kassel. The generators satisfy the commutation relations

$$[H_i, H_j] = 0, [H_i, X_j^\pm] = \pm a_{ij} X_j^\pm, [X_i^+, X_i^-] = (K_i^2 - K_i^{-2})/(s - s^{-1}),$$

where $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is the Cartan matrix for $SU(3)$ (and also the Serre

relations of degree 3 between X_1^\pm and X_2^\pm).

Comultiplication is given by

$$\begin{aligned}\Delta(H_i) &= H_i \otimes I + I \otimes H_i, \\ (\text{so } \Delta(K_i) &= K_i \otimes K_i,) \\ \Delta(X_i^\pm) &= X_i^\pm \otimes K_i + K_i^{-1} \otimes X_i^\pm,\end{aligned}$$

and the antipode S by $S(X_i^\pm) = -s^{\pm 1} X_i^\pm$, $S(H_i) = -H_i$, $S(K_i) = K_i^{-1}$.

The fundamental 3-dimensional module, which we denote by E , has a basis in which the quantum group generators are represented by the matrices Y_E as listed here.

$$\begin{aligned}X_1^+ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ X_1^- &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ H_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.\end{aligned}$$

For calculations we keep track of the elements K_i rather than H_i , represented by

$$K_1 = \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}$$

for the module E .

We can then write down the elements Y_{EE} for the actions of the generators Y on the module $E \otimes E$, from the comultiplication formulae. The R -matrix R_{EE} can be given, up to a scalar, by the prescription

$$\begin{aligned}R_{EE}(e_i \otimes e_j) &= e_j \otimes e_i, \text{ if } i > j, \\ &= s e_i \otimes e_i, \text{ if } i = j, \\ &= e_j \otimes e_i + (s - s^{-1}) e_i \otimes e_j, \text{ if } i < j,\end{aligned}$$

for basis elements $\{e_i\}$ of E .

The linear dual M^* of a module M becomes a module when the action of a generator Y on $f \in M^*$ is defined by $\langle Y_M^* f, v \rangle = \langle f, S(Y_M)v \rangle$, for $v \in M$. For the dual module $F = E^*$ we then have matrices for Y_F , relative to the dual basis, as follows.

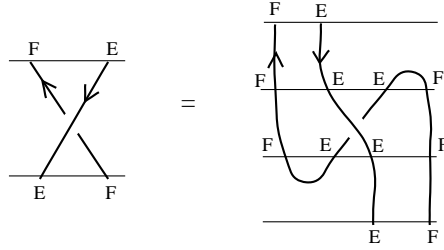
$$\begin{aligned}X_1^+ &= \begin{pmatrix} 0 & 0 & 0 \\ -s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -s & 0 \end{pmatrix} \\ X_1^- &= \begin{pmatrix} 0 & -s^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -s^{-1} \\ 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

$$K_1 = \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a \end{pmatrix}.$$

The most reliable way to work out the R -matrices R_{EF} , R_{FE} and R_{FF} is to combine R_{EE} with module homomorphisms cup_{EF} , cup_{FE} , cap_{EF} and cap_{FE} between the modules $E \otimes F$, $F \otimes E$ and the trivial 1-dimensional module, I , on which X_i^\pm acts as zero and K_i as the identity. The matrices are determined up to a scalar by such considerations; a choice for one dictates the rest.

Once these matrices have been found they can be combined with the matrix R_{EE}^{-1} to construct the R -matrices R_{EF} , R_{FE} , R_{FF} , using the diagram shown below, for example, to determine R_{EF} . This gives

$$R_{EF} = (1_F \otimes 1_E \otimes \text{cap}_{EF}) \circ (1_F \otimes R_{EE}^{-1} \otimes 1_F) \circ (\text{cup}_{FE} \otimes 1_E \otimes 1_F).$$



The module structure of $M = V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ can be found by identifying M as a 27-dimensional submodule of $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} \otimes V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$, while the two 6-dimensional modules $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ and $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ are themselves submodules of $E \otimes E$ and $F \otimes F$ respectively.

We know, by the Pieri formula, that there is a direct sum decomposition of $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} \otimes V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ as $M \oplus N$, where $M = V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ and N is the sum of the 8-dimensional module $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ and the 1-dimensional trivial module.

We first identify the module $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ as a submodule of $E \otimes E$, knowing that $E \otimes E$ is isomorphic to $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}} \otimes F$. The full twist element on the two strings both coloured by E is represented by R_{EE}^2 which acts on $E \otimes E$ as a scalar on each of the two irreducible submodules $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ and F .

Use Maple to find bases for the two eigenspaces of R_{EE}^2 . Then we can identify $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ with the 6-dimensional one, and write P and Q for the 9×6 and 9×3 matrices whose columns are these bases. The partitioned matrix $(P|Q)$ is invertible, and its inverse, found by Maple, can be written as $\begin{pmatrix} R \\ S \end{pmatrix}$, where R is a 6×9 matrix with $RP = I_6$ and $RQ = 0$.

Regard $P = \text{inj}_{M_1} EE$ as the matrix representing the inclusion of the module $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ into $E \otimes E$. Then $R = \text{proj}_{EE} M_1$ is the matrix, in the same basis, of the projection from $E \otimes E$ to $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$. For $M_1 = V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ the module generators Y_{M_1} are given by $Y_{M_1} = RY_{EE}P$, giving the explicit action of the quantum group on $V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$.

We perform a similar calculation on $F \otimes F$ to identify the module $M_2 = V_{\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}}$ and the matrices $\text{inj}_{M_2} FF$ and $\text{proj}_{FF} M_2$, giving the action of the quantum

group on $M_2 = V_{\square\square}$ in a similar way.

We use inclusion and projection further to find the four $6^2 \times 6^2$ R -matrices $R_{M_i M_j}$. For example, to construct $R_{M_1 M_2} : M_1 \otimes M_2 \rightarrow M_2 \otimes M_1$, first map $M_1 \otimes M_2$ to $E \otimes E \otimes F \otimes F$ by $\text{inj} M_1 EE \otimes \text{inj} M_2 FF$. Then construct the R -matrix crossing two strings with $E \otimes E$ and two with $F \otimes F$ as the composite of $1 \otimes R_{EF} \otimes 1$, $R_{EF} \otimes R_{FE}$ and $1 \otimes R_{FF} \otimes 1$, and finally compose with the projections $\text{proj} FF M_2 \otimes \text{proj} EE M_1$.

A similar calculation on the module $M_1 \otimes M_2$ yields the submodule $M = V_{\square\square\square}$. The full twist on two strings, one coloured by M_1 and one by M_2 , is represented by the product $R_{M_2 M_1} R_{M_1 M_2}$ and will have one 27-dimensional eigenspace M complemented by two other eigenspaces. Taking the bases of these eigenspaces in a partitioned 36×36 matrix as above will determine a 36×27 matrix $P = \text{inj} M M_1 M_2$ and a 27×36 matrix $R = \text{proj} M_1 M_2 M$. The quantum group actions $Y_{M_1 M_2}$ on the tensor product are determined by the coproduct formulae, and the actions Y_M are then given from these using P and R . These in turn give rise to the quantum group actions Y_{MM} on $M \otimes M$.

We are also able to construct the $27^2 \times 27^2$ R -matrix R_{MM} using the same inclusion and projection to map $M \otimes M$ into $M_1 \otimes M_2 \otimes M_1 \otimes M_2$, followed by the matrix for crossing four strands, built up from the R -matrices $R_{M_i M_j}$ and then the projections back to $M \otimes M$.

3.2 Completing the calculations

Remark 2. We can reach this stage directly if we know the six module generators Y_M and the R -matrix R_{MM} for the module $M = V_{\square\square\square}$. We can then calculate the module generators Y_{MM} using the coproduct, and the twisting element $T_M = (K_{1M})^4 (K_{2M})^4$.

Knowing the module generators Y_{MM} gives an immediate means of finding the highest weight vectors as common null-vectors of X_{iMM}^+ , and their weights can be identified. All the submodules of $M \otimes M$ occur with multiplicity 1 except V_ν with partition $\nu = 6, 4, 2$ whose highest weights are 2, 2. The 3-dimensional space W_ν of highest weight vectors for ν is found by solving the linear equations $X_{1MM}^+ v = 0$, $X_{2MM}^+ v = 0$, $K_{1MM} v = a^2 v$ and $K_{2MM} v = a^2 v$ for v . We then find the 2-dimensional positive eigenspace for R_{MM} on W_ν . The endomorphisms J_A and J_B will preserve this eigenspace.

Represent the 3-braid $\sigma_2 \sigma_1^{-1} \sigma_2$ in the 2-tangle A by an endomorphism F_A of $M \otimes M \otimes M$, using R_{MM} and its inverse. Then use T_M and the partial trace to close off one string, hence giving the endomorphism J_A of $M \otimes M$ determined by A . Explicitly, choose a basis $\{e_i\}$ of M and write

$$F_A(v \otimes T_M(e_i)) = \sum_j f_{ij}(v) \otimes e_j$$

with $f_{ij}(v) \in M \otimes M$. Then $J_A(v) = \sum_i f_{ii}(v)$. Applied to each of the two vectors in the highest weight space this determines a 2×2 matrix A_ν representing

the restriction of J_A to this subspace. Similarly B_ν is found using the mirror image braid $\sigma_2^{-1}\sigma_1\sigma_2^{-1}$.

We know that R_{MM} acts as a scalar on the 2-dimensional space so $J(K; V_\mu) - J(K'; V_\mu)$ is a non-zero scalar multiple of $\text{tr}(A_\nu B_\nu A_\nu B_\nu - B_\nu A_\nu A_\nu B_\nu)$.

This difference is $2(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)(q^4 + 1)(q^6 + q^3 + 1)^2(q^4 - q^2 + 1)^2(q^4 + q^3 + q^2 + q + 1)^3(q^2 + 1)^4(q^2 + q + 1)^4(q^2 - q + 1)^4(q + 1)^{10}(q - 1)^{18}$, up to a power of $q = s^2$ and the quantum dimension of V_ν .

3.3 Further examples of difference

Using the same matrices A_ν and B_ν it is possible to find further pretzel knot examples based on sequences of the tangles A and B where the 6-parallel have different Homfly polynomial, such as the knots $K(3, 3, 3, -3, -3)$ and $K(3, 3, -3, 3, -3)$. The difference here is the same as for the first example multiplied by the factor $2q^{32} - q^{31} - 3q^{30} + 5q^{29} + 3q^{28} - 10q^{27} + q^{26} + 14q^{25} - 6q^{24} - 19q^{23} + 21q^{22} + 20q^{21} - 46q^{20} + 2q^{19} + 61q^{18} - 48q^{17} - 35q^{16} + 83q^{15} - 27q^{14} - 66q^{13} + 72q^{12} + 3q^{11} - 57q^{10} + 40q^9 + 10q^8 - 33q^7 + 16q^6 + 7q^5 - 12q^4 + 7q^3 - 4q + 2$. The same calculations guarantee that satellites based on any closed 6-tangle $Q = \widehat{T}$ will have different Homfly polynomial, provided that the trace c_μ of the endomorphism $J_{\widehat{T}}$ on the highest weight space W_μ of $V^{\otimes 6}$ is non-zero, where μ is the partition 4, 2. This will be the case for most, but not all, patterns Q , and certainly will be the case for many satellites which are knots rather than links.

The calculations in section 3.2 also show that the 4-parallel of the two pretzel knots $K(1, 3, 3, -3, -3)$ and $K(1, 3, -3, 3, -3)$ with two strings oriented in one direction and two in the opposite direction will have different Homfly polynomials, by using the decomposition of the corresponding $sl(3)_q$ module $W = V \otimes V \otimes V_{\square} \otimes V_{\square}$ into a sum of irreducible $sl(3)_q$ modules. The only module to figure in this decomposition with any multiplicity in its symmetric or exterior square is again $V_{\square\square\square}$. The calculations above, using the fact that Homfly with $v = s^{-3}$ can be calculated by colouring strings with reverse orientation by the dual module V^* to the fundamental module, and that this is V_{\square} for $sl(3)_q$.

4 Cable patterns

By way of contrast, if the pattern Q is a cable on any number of strings then $K * Q$ and $K' * Q$ share the same Homfly polynomial, where K and K' have the same symmetry as in theorem 1.

Theorem 4. *Suppose that A and B are both symmetric under the half-twist τ_3 , so that*

$$A = \begin{array}{c} \boxed{A} \\ \curvearrowright \end{array}, \quad B = \begin{array}{c} \boxed{B} \\ \curvearrowright \end{array}$$

Let K and K' be knots which are the closure of ABC and BAC respectively for any tangle C , as in figure 2. Then $P(K * Q) = P(K' * Q)$ for every (m, n) cable pattern Q where m and n are coprime.

Proof. As in the proof of theorem 1 we show that $J(K * Q; V^{(N)}) = J(K' * Q; V^{(N)})$ for all N . By equation 1 it is then enough to show that $J(K; V_\mu^{(N)}) = J(K'; V_\mu^{(N)})$ for all N and all partitions $\mu \vdash m$ for which the coefficient $c_\mu \neq 0$. The coefficients c_μ depend on the pattern Q and arise as the trace of the endomorphism J_T when restricted to the highest weight space $W_\mu \subset V^{\otimes m}$, where Q is the closure of the m -braid $T = (\sigma_1 \sigma_2 \cdots \sigma_{m-1})^n$.

It is shown in [9], (see also [5]), that for any such cable Q the only non-zero coefficients c_μ occur when the partition μ is a *hook*, if m and n are coprime. It is then enough to show that $J(K; V_\mu^{(N)}) = J(K'; V_\mu^{(N)})$ for all hook partitions μ .

Using the same argument as in theorem 1 it remains to check that no Schur function s_ν occurs with multiplicity > 1 in the decomposition of either the symmetric or exterior squares, $h_2(s_\mu)$ or $e_2(s_\mu)$, for any hook partition μ . This fact has been established by Carbonara, Remmel and Yang in theorem 3 of [2], and so the proof is complete. \square

Remark 3. *Theorem 4 highlights the importance of a precise terminology for different types of satellite. The term cable is sometimes used to mean any satellite, while there is a clear distinction here between the behaviour of cables and of parallels or other satellites, which is not primarily a matter of the number of components of the satellite.*

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